The reduced effect of a single scattering with a low-mass particle via a point interaction

Jeremy Clark
jeremy.clark@fys.kuleuven.be
Katholieke Universiteit Leuven, Instituut voor Theoretische Fysica
Celestijnenlaan 200D, 3001 Heverlee, Belgium

January 15, 2009

Abstract

In this article, we study a second-order expansion for the effect induced on a large quantum particle which undergoes a single scattering with a low-mass particle via a repulsive point interaction. We give an approximation with third-order error in λ to the map $G \rightarrow \mathrm{Tr}_2[(I \otimes \rho)S_\lambda^*(G \otimes I)S_\lambda]$, where $G \in B(L^2(\mathbb{R}^n))$ is a heavy-particle observable, $\rho \in B_1(\mathbb{R}^n)$ is the density matrix corresponding to the state of the light particle, $\lambda = \frac{m}{M}$ is the mass ratio of the light particle to the heavy particle, $S_\lambda \in B(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n))$ is the scattering matrix between the two particles due to a repulsive point interaction, and the trace is over the light-particle Hilbert space. The third-order error is bounded in operator norm for dimensions one and three using a weighted operator norm on G.

1 Introduction

In theoretical physics, many derivations of decoherence models begin with an analysis of the effect on a test particle of a scattering with a single particle from a background gas [9, 6, 8]. A regime that the theorists have studied and which has generated interest in experimental physics [7] is when the test particle is much more massive than a single particle from the gas. Mathematical progress towards justifying the scattering assumption made in the physical literature in the regime where a test particle interacts with particles of comparatively low mass can be found in [1, 3, 5]. In this article, we study a scattering map expressing the effect induced on a test particle of mass M by an interaction with a particle of mass $m = \lambda M$, $\lambda \ll 1$. The force interaction between the test particle and the gas particle is taken as a repulsive point potential.

We work towards bounding the error $\epsilon(G, \lambda)$ in operator norm for $G \in B(L^2(\mathbb{R}^n))$, $n = 1, 3$ of a second order approximation:

$$\mathrm{Tr}_2[(I \otimes \rho)\mathbf{S}_\lambda^*(G \otimes I)\mathbf{S}_\lambda] = G + \lambda M_1(G) + \lambda^2 M_2(G) + \epsilon(G, \lambda), \tag{1.1}$$

where $\rho \in B_1(L^2(\mathbb{R}^n))$ is a density matrix (i.e. $\rho \geq 0$ and $\mathrm{Tr}[\rho] = 1$), $G \in B(L^2(\mathbb{R}^n))$, $\mathbf{S}_\lambda \in B(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n))$ is the unitary scattering operator for a point interaction, and the partial trace is over the second component of the Hilbert space $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$. M_1 and M_2

are linear maps acting on a dense subspace of $B(L^2(\mathbb{R}^n))$ (M_2 is unbounded). Our main result is that there exists a $c>0$ such that for all ρ, G, and $0 \le \lambda$

$$\|\epsilon(G,\lambda)\| \le c\lambda^3 \|\rho\|_{wtn}\|G\|_{wn},$$

where $\|\cdot\|_{wn}$ is a weighted operator norm of the form

$$\|G\|_{wn} = \|G\| + \||\vec{X}|G\| + \|G|\vec{X}|\| \\ + \sum_{0\le i,j\le d} (\|X_i P_j G\| + \|G P_j X_i\|) + \sum_{e_1+e_2\le 3} \||\vec{P}|^{e_1} G |\vec{P}|^{e_2}\|,$$

and $\|\cdot\|_{wtn}$ is a weighted trace norm which will depend on the dimension. In the above, $\vec{X}$ and $\vec{P}$ are the vector of position and momentum operators respectively: $(X_j f)(x) = x_j f(x)$ and $(P_j f)(x) = i(\frac{\partial}{\partial x_j} f)(x)$. Expressions of the type A^*GB for unbounded operators A and B are identified with the kernel of the densely defined quadric form $F(\psi_1;\psi_2) = \langle A\psi_1|GB\psi_2\rangle$ in the case that F is bounded.

The scattering operator is defined as $\mathbf{S}_\lambda = (\Omega^+)^*\Omega^-$, where

$$\Omega^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_{tot}} e^{-itH_{kin}} \tag{1.2}$$

are the Möller wave operators, and H_{kin} is the kinetic Hamiltonian and is the standard self-adjoint extension of the sum of the Laplacians $-\frac{1}{2M}\Delta_{heavy} - \frac{1}{2m}\Delta_{light}$, while the total Hamiltonian H_{tot} includes an additional repulsive point interaction between the particles. The definition of H_{tot} is a little tricky for $n>1$ since, in analogy to the Hamiltonian for a particle in a point potential [2], it can not be defined as a perturbation of H_{kin} even in the sense of a quadratic form. Rather, it is defined as a self-adjoint extension of $-\frac{1}{2M}\Delta_{heavy} - \frac{1}{2m}\Delta_{light}$ with a special boundary condition. Going to center of mass coordinates, we can write

$$\frac{1}{2M}\Delta_{heavy} + \frac{1}{2m}\Delta_{light} = \frac{1}{2(m+M)}\Delta_{cm} + \frac{M+m}{2mM}\Delta_{dis},$$

so that the special boundary condition will be placed on the displacement coordinate corresponding to Δ_{dis} and follows in analogy with that a single particle in a point potential as discussed in [2]. This also allows us to write down expressions for $\mathbf{S}_\lambda$. Non-trivial point potentials in dimensions >3 do not exist and the main result of our analysis is restricted to dimensions one and three.

The first and second order expressions $M_1(G)$ and $M_2(G)$ respectively have the form

$$M_1(G) = i[V_1, G] \text{ and } M_2(G) = i[V_2 + \frac{1}{2}\{\vec{A},\vec{P}\}, G] + \varphi(G) - \frac{1}{2}\varphi(I)G - \frac{1}{2}G\varphi(I), \tag{1.3}$$

where V_1, V_2, and $(\vec{A})_j$ for $j = 1,\ldots,n$ are bounded real-valued functions of the operator $\vec{X}$, and φ is a completely positive map admitting a Kraus decomposition:

$$\varphi(G) = \sum_j \int_{\mathbb{R}^3} d\vec{k}\, m^*_{j,\vec{k}}\, G\, m_{j,\vec{k}}, \tag{1.4}$$

with the $m_{j,\vec{k}}$'s being bounded multiplication operators in the $\vec{X}$-basis. Notice that terms in (1.3) are reminiscent of the form of a Lindblad generator [10]. In [4] the results of this

article are applied to the convergence of a quantum dynamical semigroup to a limiting form with generator including the terms (1.3).

The explicit forms for V_1, V_2 , $\vec{A}$, and φ are:

$$\begin{aligned} V_1 &= c_n\, s_n^{-1} \int_{\mathbb{R}^+} dk\, |k|^{-1} \int_{|\vec{v}_1|=|\vec{v}_2|=k} d\vec{v}_1\, d\vec{v}_2\, \rho(\vec{v}_1,\vec{v}_2)\, e^{i\vec{X}(\vec{v}_1-\vec{v}_2)}, & (1.5)\\ V_2 &= c_n\, s_n^{-1} \int_{\mathbb{R}^+} dk\, |k|^{-1} \int_{|\vec{v}_1|=|\vec{v}_2|=k} d\vec{v}_1\, d\vec{v}_2\, (\vec{v}_1+\vec{v}_2)\nabla_T \rho(\vec{v}_1,\vec{v}_2)\, e^{i\vec{X}(\vec{v}_1-\vec{v}_2)}, & (1.6)\\ \vec{A} &= c_n\, s_n^{-1} \int_{\mathbb{R}^+} dk\, |k|^{-1} \int_{|\vec{v}_1|=|\vec{v}_2|=k} d\vec{v}_1\, d\vec{v}_2\, \nabla_T \rho(\vec{v}_1,\vec{v}_2)\, e^{i\vec{X}(\vec{v}_1-\vec{v}_2)}, & (1.7)\\ \varphi(G) &= c_n^2\, s_n^{-2} \int_{\mathbb{R}^n} d\vec{k}\, |\vec{k}|^{-2} \int_{|\vec{v}_1|=|\vec{v}_2|=|\vec{k}|} d\vec{v}_1\, d\vec{v}_2\, \rho(\vec{v}_1,\vec{v}_2)\, e^{i\vec{X}(-\vec{v}_1+\vec{k})}\, G\, e^{-i\vec{X}(-\vec{v}_2+\vec{k})}, & (1.8) \end{aligned}$$

where s_n is surface area of a unit ball in $\mathbb{R}^n$, c_n is a constant arising form the scattering operator $\mathbf{S}_\lambda$, $\rho(\vec{k}_1,\vec{k}_2)$ is the integral kernel of ρ, and ∇_T is the gradient of weak derivatives in the diagonal direction which is formally $(\nabla_T\rho(\vec{k}_1,\vec{k}_2))_j = \lim_{h\to 0} h^{-1}\big(\rho(\vec{k}_1+h\,e_j,\vec{k}_2+h\,e_j)-\rho(\vec{k}_1,\vec{k}_2)\big)$. The integral kernel $\rho(\vec{k}_1,\vec{k}_2)$ is well defined since ρ is traceclass and hence Hilbert-Schmidt. In dimension one, the integrals $\int_{|v_1|=|v_2|=k}$ are replaced by discrete sums. In dimension two, V_2 has an additional term due to the logarithm in (4.4) which we did not write down in the expression for V_2 above. The multiplication operators $m_{j,\vec{k}}$ are defined as $m_{j,\vec{k}}(\vec{X}) = c_n s_n^{-1}\sqrt{\beta_j}\int_{|\vec{v}|=|\vec{k}|} d\vec{v}\, f_j(\vec{v})\, e^{-i\vec{X}(-\vec{v}+\vec{k})}$, where $\rho=\sum_j \beta_j |f_j\rangle\langle f_j|$ is the diagonalized form of ρ. V_1, V_2, $\vec{A}$, and φ are bounded under certain norm restrictions on ρ, since, for example, $\|V_1\|\leq c_n\||\vec{P}|^{n-2}\rho\|_1$ and $\|\varphi\| = c_n\||\vec{P}|^{n-2}\rho|\vec{P}|^{n-2}\|_1$.

With the center of mass coordinate at the origin, the scattering operator $\mathbf{S}$ (neglecting the index λ until it is explained) acts identically on the center-of-mass component of $L^2(\mathbb{R}^{2n}) = L^2(\mathbb{R}^n)\otimes L^2(\mathbb{R}^n)$ as

$$\mathbf{S} = I + I_{cm}\otimes (S(k)\otimes|\phi\rangle\langle\phi|), \tag{1.9}$$

where the right copy of $L^2(\mathbb{R}^n)$ corresponds to the displacement variable and is decomposed in the momentum basis into a radial and an angular component as $L^2(\mathbb{R}^+, r^{n-1}dr)\otimes L^2(\partial B_1(0))$, $S(k)$ acts as a multiplication operator on the $L^2(\mathbb{R}^+)$ component, and $\phi = (s_n)^{-\frac{1}{2}}1_{\partial B_1(0)}$, is the normalized indicator function over the whole surface $\partial B_1(0)$. We call $S(k)$ the scattering coefficient, and it has the form

$$\begin{aligned} &\mathbf{Dim}-1: && \mathbf{Dim}-2: && \mathbf{Dim}-3: \\ &S_\alpha(k) = \frac{-i\alpha}{k+i\frac{1}{2}\alpha} && S_l(k) = \frac{-i\pi}{l^{-1}+\gamma+\ln(\frac{k}{2})+i\frac{\pi}{2}} && S_l(k) = \frac{-2ik}{l^{-1}+ik}, \end{aligned} \tag{1.10}$$

where α is a resonance parameter defined for the one-dimensional case, l is the scattering length in the two- and three-dimensional cases and $\gamma\sim .57721$ is the Euler-Mascheroni constant. In the one-dimensional case a scattering length l is sometimes defined as the negative inverse of the resonance parameter $\alpha = \frac{\mu c}{\hbar^2}$, where c is the coupling constant of the interaction and μ is the relative mass $\frac{mM}{m+M} = M\frac{\lambda}{1+\lambda}$. However, this contrasts with the two- and three-dimensional cases where the scattering length is proportional to the strength of the interaction. In the context of this article, where the point interaction is between a light and an heavy particle,

we parameterize the resonance parameter as $\alpha = \frac{\lambda}{1+\lambda}\alpha_0$ in the one-dimensional case and the scattering length as $l = \frac{\lambda}{1+\lambda}l_0$ in the two- and three-dimensional cases for some fixed α_0 and l. This corresponds to holding the strength of the interaction fixed. Thus $\mathbf{S}_\lambda$ and S_λ will be indexed by λ for the remainder of the article.

There are two main obstacles in attempting to find a bound for the error $\epsilon(G,\lambda)$ from (1.1). The first obstacle is to find helpful expressions to facilitate making a Taylor expansion in λ of $\mathrm{Tr}_2[(I\otimes\rho)\mathbf{S}_\lambda^*(G\otimes I)\mathbf{S}_\lambda]$. Writing $\mathbf{A}_\lambda = \mathbf{S}_\lambda - I$, then

$$\mathrm{Tr}_2[(I\otimes\rho)\mathbf{S}_\lambda^*(G\otimes I)\mathbf{S}_\lambda] = G + \mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*]G + G\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda] + \mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*(G\otimes I)\mathbf{A}_\lambda],$$

and it turns out to be natural at all points of the analysis to approach the terms on the right individually. Propositions 2.2 and 2.3 are directed towards finding expressions for

$$\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*] \text{ and } \mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*(G\otimes I)\mathbf{A}_\lambda] \tag{1.11}$$

respectively (since $\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda]$ is merely the adjoint of $\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*]$). The expressions we find in Propositions 2.2 and 2.3 are of the form

$$\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*]G = \int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\, U_{\vec{k},\sigma}^* f_{\vec{k},\sigma}^*\, G, \text{ and} \tag{1.12}$$

$$\mathrm{Tr}_2[(I\otimes\rho)\mathbf{A}_\lambda^*(G\otimes I)\mathbf{A}_\lambda] = \int_{\mathbb{R}^n} d\vec{k}\int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, U_{\vec{k},\sigma_1,\lambda}^*\, h_{\vec{k},\sigma_1,\lambda}^*\, G\, h_{\vec{k},\sigma_2,\lambda}\, U_{\vec{k},\sigma_2,\lambda}, \tag{1.13}$$

for some unitaries $U_{\vec{k},\sigma}^*$, $U_{\vec{k},\sigma_2,\lambda}$ and some bounded operators $g_{\vec{k},\sigma_2,\lambda}$, $h_{\vec{k},\sigma_1,\lambda}^*$ which are functions of the vector of momentum operators $\vec{P}$. In general, we will have the problem that

$$\int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\, \|g_{\vec{k},\sigma,\lambda}\| = \infty, \text{ and } \int_{\mathbb{R}^n} d\vec{k}\int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, \|h_{\vec{k},\sigma_1,\lambda}\|\,\|h_{\vec{k},\sigma_2,\lambda}\| = \infty,$$

so the integrals of operators only have strong convergence. Propositions 3.1 and 3.3 make sense of the integrals of operators such as (1.12) and (1.13) that arise and give operator norm bounds for the limits. The basic pattern in the proof of Propositions 3.1 and 3.3 is an application of the simple inequalities in Propositions A.2 and A.3 in addition to intertwining relations that we have between the multiplication operators and the unitaries appearing in (1.12) and (1.13).

Bounding the third order error of the expansions in λ of the strongly convergent integrals (1.12) and (1.13) brings up the second major obstacle. We will need to bound certain strongly convergent integrals for all λ in a neighborhood of zero. For small λ there will be unbounded expressions arising from the scattering coefficient $S_\lambda(k)$ that will have contrasting properties between the one- and three-dimensional cases. For example, In the limit $\lambda\to 0$, $\frac{1}{\lambda}S_\lambda(k)$ becomes increasingly peaked in absolute value at $k\sim 0$ in the one-dimensional case. For the three-dimensional case, $\frac{1}{\lambda}S_\lambda(k)$ becomes increasingly peaked at $k=\infty$. A difficulty with the two-dimensional case is the presence of the natural logarithm in the expression for $S_\lambda(k)$ and the fact that $\frac{1}{\lambda}S_{\frac{\lambda}{1+\lambda}l_0}(k)$ is not peaked at a fixed point as λ varies. The peak point does tend towards $k\sim 0$ as $\lambda\to 0$, but it is unknown how to attain the necessary inequalities in this case.

This article is organized as follows. Section 2 is concerned with proving Propositions 2.2 and 2.3 which give expressions for $\mathrm{Tr}[(I\otimes\rho)\mathbf{A}_\lambda^*]$ and $\mathrm{Tr}[(I\otimes\rho)\mathbf{A}_\lambda^*(G\otimes I)\mathbf{A}_\lambda]$. In Section 3

we prove Propositions 3.1 and 3.3 which give the primary tools for bounding the integrals of operators which will arise in bounding the error term $\epsilon(G,\lambda)$ of our expansion (1.1). Section 4 contains the proof of Theorem 4.2 which is the main result of the article. This involves expanding the expressions in Propositions 2.2 and 2.3 that we found in Section 2 in λ and bounding the error. The difficult parts of the proof are characterized by using the Propositions 3.1 and 3.3 to translate unbounded expressions arising from the expansion of the scattering coefficient S_λ into conditions on G and ρ through the weighted norms $\|G\|_{wn}$ and $\|\rho\|_{wtn}$ being finite. Sections 2 and 3 apply to dimensions one through three (all dimensions where non-trivial point potentials exist), while Section 4 does not treat dimension two.

2 Finding useful expressions for a single scattering

In this section, we will find expressions for $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]$ and $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$. For notational convenience, we will begin identifying $I\otimes\rho$ with ρ and $G\otimes I$ with G. Finding formulas for $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]$, $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$ begins with writing $\mathbf{A}_\lambda=\mathbf{S}_\lambda-I$ in a convenient way. Let $f,g\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$, where the first and second component of $\mathbb{R}^n\times\mathbb{R}^n$ correspond to the displacement and the center of mass coordinate, then

$$\langle g|\mathbf{A}_\lambda f\rangle=\int_{\mathbb{R}^n}d\vec{K}_{cm}\int_0^\infty dk\,\frac{S_\lambda(k)}{s_n k^{n-1}}\Big(\int_{\partial B_k(0)}d\hat{k}_1\,\bar{g}(\hat{k}_1,\vec{K}_{cm})\Big)\Big(\int_{\partial B_k(0)}d\hat{k}_2\,f(\hat{k}_2,\vec{K}_{cm})\Big).\qquad(2.1)$$

The above formula gives a quadratic form representation of $\mathbf{A}_\lambda$ that involves integrating over a surface of $3n-1$ degrees of freedom rather than $4n$, since it acts identically over the center-of-mass component of the Hilbert space and conserves energy for the complementary displacement coordinate. The integral kernel for $\mathbf{A}_\lambda$ in center-of-mass momentum coordinates can be formally expressed as

$$\mathbf{A}_\lambda(k_{dis,1},K_{cm,1};k_{dis,2},K_{cm,2})=\frac{S_\lambda(|k_{dis,1}|)}{s_n|k_{dis,1}|^{n-1}}\delta(|k_{dis,1}|-|k_{dis,2}|)\delta(K_{cm,1}-K_{cm,2})$$

However, for instance, this does not work directly towards finding even a formal expression for

$$(\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]G)(\vec{K}_1,\vec{K}_2)=\int d\vec{k}_1\,d\vec{k}_2\,d\vec{K}\,\rho(k_1,k_2)\,\mathbf{A}_\lambda^*(\vec{k}_2,\vec{K}_1;\vec{k}_1,\vec{K})\,G(\vec{K},\vec{K}_2),\qquad(2.2)$$

where we have written down a formal equation between integral kernel entries $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G]$, G, and $\mathbf{A}_\lambda^*$ using momentum coordinates corresponding to the heavy particle and the light particle. In finding an expression for (2.2), it would be natural to have $\vec{K}_2$ as a parameterizing variable since the expression above is just multiplication of G from the left by $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]$.

For $\lambda=\frac{m}{M}$, the center of mass coordinates are $\vec{X}_{cm}=\frac{\lambda}{1+\lambda}\vec{x}+\frac{1}{1+\lambda}\vec{X}$ and $x_d=\vec{x}-\vec{X}$, where $\vec{x}$ and $\vec{X}$ are the position vectors of the particle with mass m and M. The corresponding momentum coordinates are $\vec{k}_d=\frac{1}{1+\lambda}\vec{k}-\frac{\lambda}{1+\lambda}\vec{K}$ and $\vec{K}_{cm}=\vec{k}+\vec{K}$. The proposition below gives two quadratic form representations of $\mathbf{A}_\lambda$ using different parameterisations of the integration in (2.1). (2.3) is directed towards finding an expression for $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]$ and (2.4) is for $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$. The proof of the following proposition requires changes of integration.

Proposition 2.1 (Quadratic form representations of $\mathbf{A}_\lambda$)**.** *Let $f,g\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$, then*

1. *First Quadratic Form Representation*

$$\langle g|\mathbf{A}_\lambda f\rangle = \int_{\mathbb{R}^n} d\vec{k}\, d\vec{K} \int_{SO_n} d\sigma\, S_\lambda(|\vec{k}|)$$
$$\bar{g}(\vec{k}+\lambda(\vec{K}+\sigma\vec{k}), \vec{K}+(\sigma-I)\vec{k})\, f(\sigma\vec{k}+\lambda(\vec{K}+\sigma\vec{k}), \vec{K}), \quad (2.3)$$

2. *Second Quadratic Form Representation*

$$\langle g|\mathbf{A}_{l'} f\rangle = \int d\vec{K}_2\, d\vec{k}_1 \int_{SO_n} d\sigma\, \det(I+\lambda\sigma)^{-1} S_\lambda(|\frac{I}{I+\lambda\sigma}(\vec{k}_1-\lambda\vec{K}_2)|)$$
$$\bar{g}(\vec{k}_1, \vec{K}_2+\frac{(\sigma-I)}{1+\lambda\sigma}(\vec{k}_1-\lambda\vec{K}_2))\, f(\vec{k}_1+\frac{\sigma-I}{1+\lambda\sigma}(\vec{k}_1-\lambda\vec{K}_2), \vec{K}_2), \quad (2.4)$$

where the total Haar measure on SO_n is normalized to be 1 *(and for dimension one, the integral over SO_n is replaced by a sum over $\{+,-\}$).*

The proofs of Propositions 2.2 and 2.3 work by using the spectral decomposition of ρ, special cases of G, etc. so that the quadratic form representations (2.3) and (2.4) of $\mathbf{A}_\lambda$ can be applied. Defining $\tau_{\vec{k}} = e^{i\vec{k}\cdot\vec{X}}$, recall that $\tau_{\vec{k}}$ acts in the momentum basis as a shift: $(\tau_{\vec{k}} f)(\vec{p}) = f(\vec{p}-\vec{k})$.

Proposition 2.2. *Let ρ have continuous integral operator elements in momentum representation.* $\mathrm{Tr}_2[\rho A_\lambda^*]$ *has the integral form*

$$\tilde{B}_\lambda^* = \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n} d\sigma\, \tau_{\vec{k}}\, \tau_{\sigma\vec{k}}^*\, p_{\vec{k},\sigma,\lambda}\, \bar{S}_\lambda(|\vec{k}|), \quad (2.5)$$

where $\tau_{\vec{a}}$ is a translation by $\vec{a}$ in the momentum $\vec{P}$ basis and $p_{\vec{k},\sigma,\lambda}$ is a multiplication operator:

$$p_{\vec{k},\sigma,\lambda} = \rho((1+\lambda)\vec{k}+\lambda\vec{P}, (\sigma+\lambda)\vec{k}+\lambda\vec{P}).$$

Proof. The following equality holds:

$$\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*] = \mathrm{Tr}_2[\sum_j \beta_j|f_j\rangle\langle f_j|\mathbf{A}_\lambda^*] = \sum_j \beta_j(\mathbf{id}\otimes\langle f_j|)\mathbf{A}_\lambda^*(\mathbf{id}\otimes|f_j\rangle),$$

where the infinite sum on the right converges absolutely in the operator norm. If we take a partial sum $\rho_m = \sum_{j=1}^m \beta_j|f_j\rangle\langle f_j|$, then using (2.3),

$$\sum_{j=1}^m \langle w|(\mathbf{id}\otimes\langle f_j|)\mathbf{A}_\lambda^*(\mathbf{id}\otimes|f_j\rangle)v\rangle = \sum_{j=1}^m \int_{\mathbb{R}^n\times\mathbb{R}^n} d\vec{K}_1 d\vec{K}_2 \int d\vec{k}\, \bar{S}_\lambda(|\vec{k}|) \int d\sigma$$
$$\bar{f}_j(\sigma\vec{k}+\lambda(\vec{K}_1+\sigma\vec{k}))\, \bar{w}(\vec{K}_1)\, f_j(\vec{k}+\lambda(\vec{K}_1+\sigma\vec{k}))\, v(\vec{K}_2+(\sigma-I)\vec{k}).$$

This has the form $\langle w|[\cdot]v\rangle$, where $[\cdot]$ is given by

$$\int_{\mathbb{R}^n} d\vec{k}\bar{S}_\lambda(\vec{k}) \int_{SO_n} d\sigma\, \tau_{\sigma\vec{k}}^*\, \tau_{\vec{k}}\, \rho_m((1+\lambda)\vec{k}+\lambda\vec{K}, (\sigma+\lambda)\vec{k}+\lambda\vec{K}).$$

This converges in operator norm to the expression given by (2.5), since $\rho_m \to \rho$ in the trace norm and by the bound given in Corollary 3.2.

∎

$\mathrm{Tr}_2[\rho\mathbf{A}_\lambda]$ has a similar integral representation by taking the adjoint. Now we will delve into the form of $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$. In the following, the operator D_A acts on $f \in L^2(\mathbb{R}^n)$ as $(D_A f)(\vec{k}) = |\det(A)|^{\frac{1}{2}} f(A\vec{k})$ for a element $A \in GL_n(\mathbb{R})$.

Proposition 2.3. *Let $\sum_j \beta_j |f_j\rangle\langle f_j|$ be the spectral decomposition of ρ. $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$ can be written in the form*

$$\tilde{\mathbf{B}}_\lambda(G) = \sum_j \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1 d\sigma_2\, U^*_{\vec{k},\sigma_1,\lambda}\, m^*_{j,\vec{k},\sigma_1,\lambda}\, \bar{S}_\lambda\big(\big|\frac{\vec{k}-\lambda\vec{P}}{1+\lambda}\big|\big)$$
$$G\, S_\lambda\big(\big|\frac{\vec{k}-\lambda\vec{P}}{1+\lambda}\big|\big)\, m_{j,\vec{k},\sigma_2,\lambda}\, U_{\vec{k},\sigma_2,\lambda}, \quad (2.6)$$

where $U_{\vec{k},\sigma_2,\lambda} = \tau^*_{\vec{k}} D_{\frac{1+\lambda\sigma}{1+\lambda}} \tau_{\sigma\vec{k}},$. $\tau_{\sigma k}$, $\tau_{\vec{k}}$, and $D_{\frac{1+\lambda}{1+\lambda\sigma}}$ act on the momentum basis and $m_{j,\vec{k},\sigma,\lambda}$ is a function of the momentum operator $\vec{P}$ of the form

$$\sqrt{\beta_j}\det(1+\lambda\sigma)^{-\frac{1}{2}} f_j\big(\vec{k} + \frac{\sigma - I}{I+\lambda}(\vec{k}-\lambda\vec{P})\big).$$

Proof. Equation (2.4) tells us how $\mathbf{A}_\lambda$ acts as a quadratic form. In order to use (2.4), we will look at $\langle v|\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]w\rangle$ in the special case where $G = G\otimes I = |y\rangle\langle y|\otimes I$ is a one-dimensional projection tensored with the identity over the light-particle Hilbert space. Formally, this allows us to write

$$\langle v|\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]w\rangle = \sum_j\sum_l \beta_j \langle v\otimes f_j|\mathbf{A}_\lambda^*|y\otimes\phi_l\rangle\langle y\otimes\phi_l|\mathbf{A}_\lambda|w\otimes f_j\rangle,$$

where (ϕ_m) is some orthonormal basis over the light-particle Hilbert space allowing a representation of the identity operator as a sum of one-dimensional projections , and the spectral decomposition of ρ has been used. Once (2.4) has been applied, we build up to an expression (2.6), taking care with respect to the limits involved. By Corollary 3.4, the expression (2.6) defines a bounded completely positive map (c.p.m.). Since $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$ defines a c.p.m. and agrees with (2.6) for one-dimensional orthogonal projections, it follows that the two expressions are equal on $B(L^2(\mathbb{R}^d))$. This follows because c.p.m.'s are strongly continuous and the span of one-dimensional orthogonal projections is strongly dense.

The following holds, where the right-hand side converges in the operator norm:

$$\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda] = \sum_j \beta_j(\mathbf{id}\otimes\langle f_j|)\mathbf{A}_\lambda^* G\mathbf{A}_\lambda(\mathbf{id}\otimes|f_j\rangle).$$

For $G = |y\rangle\langle y|$, $(\mathbf{id}\otimes\langle f_j|)\mathbf{A}_\lambda^* G\mathbf{A}_\lambda(\mathbf{id}\otimes|f_j\rangle) = \varphi_{y,j}(I)$, where $\varphi_{y,j}$ is the completely positive map such that for $H \in B(\mathcal{H})$

$$\varphi_{y,j}(H) = (\mathbf{id}\otimes\langle f_j|)\mathbf{A}_\lambda^*(|y\rangle\langle y|\otimes H)\mathbf{A}_\lambda(\mathbf{id}\otimes|f_j\rangle).$$

Since $\varphi_{y,j}$ is completely positive, $\varphi_{y,j}(\sum_{l=1}^m |\phi_l\rangle\langle\phi_l|)$ converges strongly to $\varphi_{y,j}(I)$. $\varphi_{y,j}(I)$ is determined by its expectations $\langle v|\varphi_y(I)v\rangle$, and moreover

$$\langle v|\varphi_{y,j}(I)v\rangle = \lim_{N\to\infty}\langle v|\varphi_{y,j}(\sum_{m=1}^N |\phi_m\rangle\langle\phi_m|)v\rangle$$
$$= \lim_{N\to\infty}\sum_{m=1}^N \langle\phi_m|\upsilon_{v,j,y}\rangle\langle\upsilon_{v,j,y}|\phi_m\rangle = \|\upsilon_{v,j,y}\|^2 = \int_{\mathbb{R}^n} d\vec{k}\,\bar{\upsilon}_{v,j,y}(\vec{k})\,\upsilon_{v,j,y}(\vec{k}), \quad (2.7)$$

where $\upsilon_{v,j,y}$ is defined as the vector $\upsilon_{v,j,y} = ((\langle y| \otimes \mathbf{id})\mathbf{A}_\lambda(|v\rangle \otimes |f_j\rangle)$. Using (2.4), $\langle \phi_m | \upsilon_{v,j,y}\rangle$ can be expressed as

$$\langle \phi_m | \upsilon_{v,j,y}\rangle = \int d\vec{K} \int_{SO_n} d\sigma \, S_\lambda\big(\big|\frac{I}{I+\lambda\sigma}(\vec{k}-\lambda\vec{K})\big|\big) \, \det(I+\lambda\sigma)^{-1}$$
$$\bar{\phi}_m(\vec{k}) \, \bar{y}\big(\vec{K} + \frac{\sigma - I}{I+\lambda\sigma}(\vec{k}-\lambda\vec{K})\big) \, f_j\big(\vec{k} + \frac{\sigma - I}{I+\lambda\sigma}(\vec{k}-\lambda\vec{K}\big) \, v(\vec{K}). \quad (2.8)$$

By (2.7), we can evaluate $\mathrm{Tr}_2[(|f_j\rangle\langle f_j|)\mathbf{A}_\lambda^*(|y\rangle\langle y| \otimes I)\mathbf{A}_\lambda] = \langle v|\varphi_y(I)v\rangle$ through expression $\int_{\mathbb{R}^n} d\vec{k}\bar{\upsilon}_{v,j,y}(\vec{k})\upsilon_{v,j,y}(\vec{k})$. Through (2.8) we have an a.e. defined expression for the values $\upsilon_{v,j,y}(\vec{k})$. Now, writing down $\int_{\mathbb{R}^n} d\vec{k}\bar{\upsilon}_{v,j,y}(\vec{k})\upsilon_{v,j,y}(\vec{k})$ using the expression for $\upsilon_{v,j,y}(\vec{k})$, the result can be viewed as an integral of operators acting from the left and the right on $|y\rangle\langle y|$, followed by an evaluation $\langle v|(\cdot)v\rangle$. Using the intertwining relation:

$$m(\vec{P}) \, \tau_{\vec{k}}^* \, D_{\frac{1+\lambda\sigma}{1+\lambda}} \, \tau_{\sigma\vec{k}} = \tau_{\vec{k}}^* \, D_{\frac{1+\lambda\sigma}{1+\lambda}} \, \tau_{\sigma\vec{k}} \, m(\vec{P} - \frac{\sigma - I}{1+\lambda\sigma}(\vec{k}-\lambda\vec{P})),$$

for a function $m(\vec{P})$ of the momentum operators $\vec{P}$ and the fact that $\frac{\sigma+\lambda}{I+\lambda\sigma} = \sigma\frac{I+\lambda\sigma^{-1}}{I+\lambda\sigma}$ is an isometry for $0 \leq \lambda < 1$, the expression can be written:

$$\langle v|\varphi_{y,j}(I)v\rangle = \langle v| \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1 \, d\sigma_2 \, [U_{\vec{k},\sigma_1,\lambda}^* \, m_{j,\vec{k},\sigma_1,\lambda}^* \, \bar{S}_\lambda\big(\big|\frac{\vec{k}-\lambda\vec{P}}{1+\lambda}\big|\big)$$
$$(|y\rangle\langle y|) \, S_\lambda\big(\big|\frac{\vec{k}-\lambda\vec{P}}{1+\lambda}\big|\big) \, m_{j,\vec{k},\sigma_2,\lambda} \, U_{\vec{k},\sigma_2,\lambda}]|v\rangle.$$

So $\varphi_{y,j}(I) = \mathrm{Tr}_2[(|f_j\rangle\langle f_j|)\mathbf{A}_\lambda^*(|v\rangle\langle v|)\mathbf{A}_\lambda]$ agrees with the expression (2.6) for a fixed j and for $G = |v\rangle\langle v|$ for all v, and hence by our observation at the beginning of the proof, $\mathrm{Tr}_2[(|f_j\rangle\langle f_j|)\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$ is equal to the expression (2.6) for a single fixed j and all $G \in B(L^2(\mathbb{R}^n))$. However, if we take the limit $m \to \infty$ for $\rho_m = \sum_{j=1}^m \beta_j |f_j\rangle\langle f_j|$, then the expression (2.6) converges in the operator norm and $\mathrm{Tr}_2[\rho_m\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$ converges to $\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}_\lambda]$. Hence we have equality for all trace class ρ.

□

Through the formula $\mathrm{Tr}_2[\rho\mathbf{S}_\lambda^* G\mathbf{S}_\lambda] = G + \tilde{B}^*G + G\tilde{B} + \tilde{\mathbf{B}}(G)$, it is clear that $\tilde{B}^* + \tilde{B} = -\tilde{\mathbf{B}}(I)$ by plugging in $G = I$. However, it is not at all obvious that this equality takes place through the expressions (2.5) and (2.6) for $\tilde{B}^*$ and $\tilde{\mathbf{B}}(I)$, respectively, since the operators $U_{\vec{k},\sigma,\lambda}$ appear only in form for $\tilde{\mathbf{B}}(I)$.

It is convenient to notice the intertwining relation $h(\vec{k}-\lambda\vec{P}) \, U_{\vec{k},\sigma,\lambda} = U_{\vec{k},\sigma,\lambda} \, h(\frac{1+\lambda}{I+\lambda\sigma}\vec{k} - \lambda\vec{P}))$. Let $g \in L^2(\mathbb{R}^n)$, then $\hat{g} = \tilde{\mathbf{B}}(I)g$ can be written:

$$\hat{g}(\vec{p}) = \sum_j \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1 \, d\sigma_2 \, \big(U_{\vec{k},\sigma_1,\lambda}^* \, m_{j,\vec{k},\sigma_1,\lambda}^* \, U_{\vec{k},\sigma_1,\lambda}\big)(\vec{p})$$
$$|S_\lambda|^2 \big(\big|\frac{I}{I+\lambda\sigma_1}(\vec{k}-\lambda\vec{p})\big|\big) \, \big(U_{\vec{k},\sigma_1,\lambda}^* \, m_{j,\vec{k},\sigma_2,\lambda} \, U_{\vec{k},\sigma_1,\lambda}\big)(\vec{p}) \, (U_{\vec{k},\sigma_1,\lambda}^* U_{\vec{k},\sigma_1,\lambda} g)(\vec{p}), \quad (2.9)$$

where we have intertwined $U_{\vec{k},\sigma_1,\lambda}^*$ from the left to the right, and

$$\big(U_{\vec{k},\sigma_1,\lambda}^* \, m_{j,\vec{k},\sigma_1,\lambda}^* \, U_{\vec{k},\sigma_1,\lambda}\big)(\vec{p}) = \sqrt{\beta_j} \, \det(I+\lambda\sigma_1)^{-\frac{1}{2}} \, \bar{f}_j\big(\vec{k} + \frac{\sigma_1 - I}{I+\lambda\sigma_1}(\vec{k}-\lambda\vec{p})\big),$$

$$\big(U^*_{\vec{k},\sigma_1,\lambda}\, m_{j,\vec{k},\sigma_2,\lambda}\, U_{\vec{k},\sigma_1,\lambda}\big)(\vec{p}) = \sqrt{\beta_j}\, \det(I+\lambda\sigma_2)^{-\frac{1}{2}}\, f_j\big(\vec{k} + (\sigma_2 - I)(I+\lambda\sigma_1)^{-1}(\vec{k}-\lambda\vec{p})\big),$$

$$\big(U^*_{\vec{k},\sigma_1,\lambda}\, U_{\vec{k},\sigma_1,\lambda} g\big)(\vec{p}) = \det\big(\frac{1+\lambda}{I+\lambda\sigma_1}\big)^{\frac{1}{2}}\, \det\big(\frac{I+\lambda\sigma_2}{1+\lambda}\big)^{\frac{1}{2}}\, g(\vec{p} + (\sigma_1-\sigma_2)(I+\lambda\sigma_1)^{-1}(\vec{k}-\lambda\vec{p})).$$

Making the change of variables $\frac{\sigma_1}{I+\lambda\sigma_1}(\vec{k}-\lambda\vec{p}) \to \vec{k}$, the resulting expression has only angular dependance of $\sigma_2\sigma_1^{-1} = \sigma$, and integrating out the other angular degrees of freedom yields $\tilde{\mathbf{B}}$.

3 Bounding integrals of non-commuting operators

Now we move on to proving Propositions 3.1 and 3.3 below which are proved in much greater generality than needed for this section, but they will serve as the principle tools in Section 4. To state these propositions we will need to generalize the concept of a multiplication operator. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. Given a bounded function $M : \mathbb{R}^n \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we can construct an element $\mathbf{M} \in \mathcal{B}(L^2(\mathbb{R}^n) \otimes \mathcal{H}_1, L^2(\mathbb{R}^n) \otimes \mathcal{H}_2)$ using the equivalence $L^2(\mathbb{R}^n) \otimes \mathcal{H}_1 \cong L^2(\mathbb{R}^n, \mathcal{H}_1)$, where for $\mathbf{f} \in L^2(\mathbb{R}^n) \otimes \mathcal{H}_1$

$$\mathbf{M}(\mathbf{f})(\vec{x}) = M(\vec{x})\mathbf{f}(\vec{x}).$$

We will call these multiplication operators.

Proposition 3.1. *Define $B : L^2(\mathbb{R}^n) \otimes \mathcal{H}_1 \to L^2(\mathbb{R}^n) \otimes \mathcal{H}_2$, s.t.*

$$B = \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n} d\sigma\, \tau^*_{\vec{k}}\, \tau_{\mathbf{a}_\sigma \vec{k}}\, q_{\vec{k},\sigma}, \tag{3.1}$$

where $q_{\vec{k},\sigma}$ is a multiplication operator in the $\vec{P}$ basis of the form:

$$q_{\vec{k},\sigma} = n_{\vec{k},\sigma}(\vec{P})\eta(\mathbf{x}_{1,\sigma}\vec{k} + \mathbf{y}_\sigma \vec{P}, \mathbf{x}_{2,\sigma}\vec{k} + \mathbf{y}_\sigma \vec{P}),$$

where $\eta(\vec{k}_1, \vec{k}_2)$ is continuous and defines a trace class integral operator on $L^2(\mathbb{R}^n)$, $\mathbf{a}_\sigma, \mathbf{x}_{1,\sigma}, \mathbf{x}_{2,\sigma}, \mathbf{y}_\sigma \in M_n(\mathbb{R})$, and $n_{\vec{k},\sigma} \in \mathcal{B}(L^2(\mathbb{R}^n) \otimes \mathcal{H}_1, L^2(\mathbb{R}^n) \otimes \mathcal{H}_2)$ is a multiplication operator. Let

$$|\det(\mathbf{x}_{1,\sigma} + \mathbf{y}_\sigma(\mathbf{a}_\sigma - I))|, |\det(\mathbf{x}_{2,\sigma} + \mathbf{y}_\sigma(\mathbf{a}_\sigma - I))|, |\det(\mathbf{x}_{1,\sigma})|, \ \textit{and}\ |\det(\mathbf{x}_{2,\sigma})|$$

be uniformly bounded from below by $\frac{1}{c}$ for some $c > 0$. Finally, let the family of maps $n_{\vec{k},\sigma}(\vec{K}) \in B(\mathcal{H}_1, \mathcal{H}_2)$ satisfy the norm bound:

$$\sup_{\vec{k},\sigma} \|n_{\vec{k},\sigma}\| \le r.$$

Then B is well defined as a strong limit and is bounded in operator norm by

$$\|B\| \le cr\|\eta\|_1.$$

Proof. We check the conditions for Proposition A.2 (applied for integrals rather than sums). Due to the intertwining relations between the unitaries $\tau^*_{\vec{k}}\tau_{\mathbf{a}_\sigma \vec{k}}$ and the multiplication operators $q_{\vec{k},\sigma}$, we will then have a bound from above by an integral of multiplication operators. We must show that $\frac{1}{2}(G_1 + G_2)$ is bounded, where

$$G_1 = \int d\vec{k} \int_{SO_n} d\sigma\, |\tau^*_{\vec{k}}\, \tau_{\mathbf{a}_\sigma \vec{k}}\, q_{\vec{k}_2,\sigma_2}| \text{ and } G_2 = \int d\vec{k} \int_{SO_n} d\sigma\, |q^*_{\vec{k}_1,\sigma_1}\, \tau^*_{\mathbf{a}_\sigma \vec{k}}\, \tau_{\vec{k}}|.$$

The integrand of G_1 is the multiplication operator

$$|\tau^*_{\vec{k}}\tau_{\mathbf{a}\vec{k}}q_{\vec{k},\sigma}| = |n_{\vec{k},\sigma}(\vec{P})||\eta(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P},\mathbf{x}_{2,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P})|.$$

and the integrand of G_2 is

$$\begin{aligned}|q^*_{\vec{k},\sigma}\,\tau^*_{\mathbf{a}_\sigma\vec{k}}\,\tau_{\vec{k}}| &= \tau^*_{\vec{k}}\,\tau^*_{\mathbf{a}_\sigma\vec{k}}\,|n_{\vec{k},\sigma}(\vec{P})||\eta(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P},\mathbf{x}_{2,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P})|\,\tau^*_{\mathbf{a}_\sigma\vec{k}}\,\tau_{\vec{k}}\\ &= |n_{\vec{k},\sigma}(\vec{P}+\sigma\vec{k}-\vec{k})||\eta(\mathbf{x}'_{1,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P},\mathbf{x}'_{2,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P})| \quad (3.2)\end{aligned}$$

where $\mathbf{x}'_{j,\sigma}=\mathbf{x}_{j,\sigma}+\mathbf{y}_\sigma(\mathbf{a}_\sigma-I)$, and we have used that $\tau_k M(\vec{P})=M(\vec{P}-k)\tau_k$.

However since the operators in the integrand of G_1 are all multiplication operators in $\vec{P}$, bounding a sum on them in the operator norm can be computed as a supremum in the following way:

$$\begin{aligned}\|G_1\| &\le \sup_{\vec{P}}\|\int d\vec{k}\int_{SO_n}d\sigma\,|n_{\vec{k},\sigma}(\vec{P})||\eta(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P},\mathbf{x}_{2,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P})|\|_{B(\mathcal{H}_1)}\\ &\le \big(\sup_{\vec{P}}\|n_{\vec{k},\sigma}(\vec{P})\|_{B(\mathcal{H}_1)}\big)\sup_{\vec{P}}\big(\int d\vec{k}\int_{SO_n}d\sigma\,|\eta(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P},\mathbf{x}_{2,\sigma}\vec{k}+\mathbf{y}_\sigma\vec{P})|\big) \quad (3.3)\end{aligned}$$

A similar result holds for G_2. Now applying Lemma A.1 to (3.3) along with our conditions on $\mathbf{x}_{1,\sigma}$, $\mathbf{x}_{2,\sigma}$, and $n_{\vec{k},\sigma}(\vec{P})$ we get the bound $\|G_1\|\le rc\|\eta\|_1$.

□

Corollary 3.2. *The integral of operators (2.5) converges strongly to a bounded operator with norm less than or equal to* $\frac{1}{(1-\lambda)^n}\|\rho\|_1$.

The bound in the above corollary in not sharp, since in Proposition (2.2) we show that $\tilde{B}=\mathrm{Tr}_2[\rho\mathbf{A}^*_\lambda]$. Thus $\|\tilde{B}\|\le\|\rho\|_1\|\mathbf{S}_\lambda-I\|\le 2\|\rho\|$, since $\mathbf{S}_\lambda$ is unitary.

Proof. We apply Proposition (3.1) with $n_{\vec{k},\sigma}(\vec{P})=S_\lambda(|\vec{k}|)$, $\eta=\rho$, $\mathbf{a}_\sigma=\sigma$, $\mathbf{x}_{1,\sigma}=1+\lambda$, $\mathbf{x}_{2,\sigma}=I+\sigma$, and $\mathbf{y}_\sigma=\lambda$. $|n_{\vec{k},\sigma}(\vec{P})|\le 1$, so we can take $r=1$. All determinants involved are of operators of the form $\sigma_1+\lambda\sigma_2$ where $\sigma_1,\sigma_2\in SO_n$, so these determinants have a lower bound of $(1-\lambda)^n$. Hence we can take $c=(1-\lambda)^{-n}$.

□

Proposition 3.3. *Let* $G\in\mathcal{B}(\mathcal{H}_l\otimes L^2(\mathbb{R}^n),\mathcal{H}_r\otimes L^2(\mathbb{R}^n))$*, and* $\varphi:\mathcal{B}(\mathcal{H}_l\otimes L^2(\mathbb{R}^n),\mathcal{H}_r\otimes L^2(\mathbb{R}^n))\to\mathcal{B}(\mathcal{H}^0_l\otimes L^2(\mathbb{R}^n),\mathcal{H}^0_r\otimes L^2(\mathbb{R}^n))$ *has the form*

$$\varphi(G)=\sum_j\int d\vec{k}\int_{SO_n\times SO_n}d\sigma_1 d\sigma_2\,U^*_{\vec{k},\sigma_1}\,h^*_{j,\vec{k},\sigma_1}\,G\,g_{j,\vec{k},\sigma_2}\,U_{\vec{k},\sigma_2},$$

where $U_{\vec{k},\sigma}$ *acts on the* $L^2(\mathbb{R}^n)$ *tensor as* $U_{\vec{k},\sigma}=\tau_{\vec{k}}\,D_{\mathbf{b}_\sigma}\,\tau^*_{\mathbf{a}_\sigma\vec{k}}$*, and* $h_{j,\vec{k},\sigma}$ *and* $g_{j,\vec{k},\sigma}$ *are multiplication operators in* $\vec{P}$ *of the form:*

$$h_{j,\vec{k},\sigma}=n^{(1)}_{j,\vec{k},\sigma}(\vec{P})\eta^{(1)}_j(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{x}_{2,\sigma}\vec{P}),\ \ \textit{and}\ g_{j,\vec{k},\sigma}=n^{(2)}_{j,\vec{k},\sigma}(\vec{P})\eta^{(2)}_j(\mathbf{x}_{1,\sigma}\vec{k}+\mathbf{x}_{2,\sigma}\vec{P}).$$

In the above, $\mathbf{x}_{1,\sigma}, \mathbf{x}_{2,\sigma}, \mathbf{a}_\sigma \in M_n(\mathbb{R})$, $\mathbf{b}_\sigma \in GL_n(\mathbb{R})$, the family of operators $n^{(1)}_{j,\vec{k},\sigma}$ and $n^{(2)}_{j,\vec{k},\sigma}$ lie in $\mathcal{B}(\mathcal{H}_l, \mathcal{H}_l^0)$ and $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_r^0)$, respectively, and finally $\eta_j^{(1)}, \eta_j^{(2)} \in L^2(\mathbb{R}^n)$. We will require that

$$\inf_\sigma |\det(\mathbf{x}_{1,\sigma} + \mathbf{x}_{2,\sigma}(\mathbf{b}_\sigma^{-1}\mathbf{a}_\sigma - I))| \geq \frac{1}{c}$$

and $\sup_{j,\vec{k},\sigma} \|n^{(1)}_{j,\vec{k},\sigma}\|, \sup_{j,\vec{k},\sigma} \|n^{(2)}_{j,\vec{k},\sigma}\| \leq r$.

In this case, the integral of operator converges strongly to an operator $\varphi(G)$ with the norm bound

$$\|\varphi(G)\| \leq cr^2 \frac{1}{2}(\|T_1\|_1 + \|T_2\|_1)\|G\|.$$

where $T_\epsilon = \sum_j |\eta_j^{(\epsilon)}\rangle\langle\eta_j^{(\epsilon)}|$ for $\epsilon = 1, 2$ and $\|\cdot\|_1$ is the trace norm.

Proof. We work towards showing the conditions of Proposition A.3 with sums replaced an integral-sum. We thus need to find bounds for the operator norms of

$$\sum_j \int_{SO_n} d\sigma \int_{\mathbb{R}^n} d\vec{k}\, U^*_{\vec{k},\sigma}\, |g|^2_{j,\vec{k},\sigma}\, U_{\vec{k},\sigma} \text{ and } \sum_j \int_{SO_n} d\sigma \int_{\mathbb{R}^n} d\vec{k}\, U^*_{\vec{k},\sigma}\, |h|^2_{j,\vec{k},\sigma}\, U_{\vec{k},\sigma}. \quad (3.4)$$

$|g|^2_{j,\vec{k},\sigma} = |g|^2_{j,\vec{k},\sigma}(\vec{P})$ is a multiplication operator with elements in $\mathcal{B}(\mathcal{H}_l^0, \mathcal{H}_l^0)$ or an element in $\mathcal{B}(L^2(\mathbb{R}^n) \otimes \mathcal{H}_l, L^2(\mathbb{R}^n) \otimes \mathcal{H}_l)$. Conjugating with U_σ, we get only multiplication operators back:

$$U^*_{\vec{k},\sigma}\, |g|^2_{j,\vec{k},\sigma}(\vec{P})\, U_{\vec{k},\sigma} = |g|^2_{j,\vec{k},\sigma}(\mathbf{b}_\sigma^{-1}\vec{P} + (\mathbf{b}_\sigma^{-1}\mathbf{a}_\sigma - I)\vec{k}).$$

With the calculations for bounding integrals of multiplication operators as in the proof of (3.1), we get the bound

$$\|\varphi(G)\| \leq \frac{1}{2} c\, r^2 \sum_j (\|\eta_j^{(1)}\|_2^2 + \|\eta_j^{(2)}\|_2^2)\|G\| = \frac{1}{2} c\, r^2 (\|T_1\|_1 + \|T_2\|_1)\|G\|.$$

□

Corollary 3.4. *The integral of operators (2.6) converges strongly to a limit with operator norm bounded by $\|\rho\|_1 \|G\| \left(\frac{1}{1-\lambda}\right)^n$.*

Proof. We apply Proposition 3.3 in the case where $\mathbf{a}_\sigma = \sigma$, $\mathbf{b}_\sigma = \frac{I+\lambda\sigma}{1+\lambda}$, $\mathbf{x}_{1,\sigma} = \frac{\sigma+\lambda}{1+\lambda}$, $\mathbf{x}_{2,\sigma} = \frac{\lambda(I-\sigma)}{I+\lambda}$, $\eta_j^{(1)} = \eta_j^{(2)} = \sqrt{\beta_j} f_j$, and

$$n^{(1)}_{j,\vec{k},\sigma}(\vec{P}) = n^{(2)}_{j,\vec{k},\sigma}(\vec{P}) = \det(1 + \lambda\sigma)^{-\frac{1}{2}}.$$

In this case $|n^{(1)}_{j,\vec{k},\sigma}(\vec{P})|$ and $|n^{(2)}_{j,\vec{k},\sigma}(\vec{P})| \leq (1-\lambda)^{-\frac{n}{2}}$, so we can take $r = 1$. Also $\mathbf{x}_{1,\sigma} + \mathbf{x}_{2,\sigma}(\mathbf{b}_\sigma^{-1}\mathbf{a}_\sigma - I) = \frac{\sigma(1+\lambda)}{I+\lambda\sigma}$ and $\|(\frac{\sigma(1+\lambda)}{I+\lambda\sigma})^{-1}\| \leq 1$, and hence $|\det(\frac{\sigma(1+\lambda)}{I+\lambda\sigma})| \geq (1)^n = 1$ independent of λ and σ, so we can take $c = 1$. Hence by (3.3), we have our conclusion with a bound $\|\rho\|_1 \|G\|(1-\lambda)^{-n}$.

□

4 Reduced Born approximation with third-order error

In this section, we will prove Theorem 4.2. To make mathematical expression more compact it will be helpful to have the dictionary below. In the following expressions $\lambda, r \in \mathbb{R}^+$, $\sigma \in SO_n$, and $\vec{k}, \vec{P} \in \mathbb{R}^n$.

Dictionary of vectors in $\mathbb{R}^n$

1. $\vec{a}_{\vec{k},r,\lambda}(\vec{P}) = (1 + r\lambda)\vec{k} + r\lambda\vec{P}$
2. $\vec{v}_{\vec{k},\sigma,\lambda}(\vec{P}) = \frac{\sigma+\lambda}{1+\lambda}\vec{k} - \lambda\frac{\sigma-I}{1+\lambda}\vec{P}$
3. $\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P}) = \frac{\sigma(1+\lambda)-\lambda r(\sigma-I)}{1+\lambda}\vec{k} - \lambda r\frac{\sigma-I}{1+\lambda}\vec{P}$
4. $\vec{d}_{\vec{k},\lambda}(\vec{P}) = \frac{1}{1+\lambda}\vec{k} - \frac{\lambda}{1+\lambda}\vec{P}$

Dictionary of matrices in $M_n(\mathbb{R})$

1. $\mathbf{c}_{1,\sigma,r,\lambda} = \frac{\sigma(1+\lambda)}{\sigma(1+\lambda)-\lambda r(\sigma-I)}$
2. $\mathbf{c}_{2,\sigma,r,\lambda} = \frac{-\lambda r(1-\lambda)}{\sigma(1+\lambda)-\lambda r(\sigma-I)}$
3. $\mathbf{c}_{3,\sigma,r,\lambda} = \frac{(1+\lambda)(r+(1-r)\sigma)}{\sigma(1+\lambda)-\lambda r(\sigma-I)}$

Now we will list some relations between the vectors. The significance of these relations will become apparent once we begin doing calculations.

Relations

R1. $\vec{k} + \vec{P} = \frac{1}{1+r\lambda}\vec{a}_{\vec{k},r,\lambda}(\vec{P}) + \frac{1}{1+r\lambda}\vec{P}$

R2. $\vec{d}_{\vec{k},\lambda}(\vec{P}) = \mathbf{c}_{1,\sigma,r,\lambda}\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P}) - \lambda\mathbf{c}_{3,\sigma,r,\lambda}\vec{P}$

R3. $\vec{k} + \vec{P} = \mathbf{c}_{1,\sigma,r,\lambda}\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P}) + \mathbf{c}_{2,\sigma,r,\lambda}\vec{P}$

In the proof of (4.2) the analysis is organized around the fact that certain expressions are bounded. In the limit $\lambda \to 0$, expressions of the type $\frac{1}{\lambda}\bar{S}_\lambda(\cdot)$ will be a source of unboundedness, and ρ and G will have to be constrained in such a way as to compensate for this. The following expressions, defined for dimensions $n = 1, 3$, are uniformly bounded in $P, k \in \mathbb{R}$, $\sigma \in \{+, -\}$, $0 \le r \le 1$, and $0 \le \lambda$:

$$E_1(\vec{P}, \vec{k}, r, \lambda) = \frac{(\delta_{n,3} + |\vec{a}_{\vec{k},r,\lambda}(\vec{P})|^{n-2})^{-1}}{1 + |\vec{P}|}\frac{1}{\lambda}\bar{S}_\lambda(|\vec{k}|), \tag{4.1}$$

$$E_2(\vec{P}, \vec{k}, \sigma, r, \lambda) = \frac{(\delta_{n,3} + |\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P})|^{n-2})^{-1}}{1 + |\vec{P}|}\frac{1}{\lambda}\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|), \tag{4.2}$$

$$E_3(\vec{P}, \vec{k}, \lambda) = \frac{(\delta_{n,3} + |\vec{k}|^{n-2})^{-1}}{1 + |\vec{P}|}\frac{1}{\lambda}\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|). \tag{4.3}$$

Their boundedness can be seen by using (R1) to rewrite $\vec{k}$ in terms of $\vec{a}_{\vec{k},r,\lambda}(\vec{P})$ and $\vec{P}$ for $E_1(\vec{P}, \vec{k}, r, \lambda)$, (R2) to write $\vec{d}_{\vec{k},\lambda}(\vec{P})$ in terms of $v_{k,\sigma,r,\lambda}(P)$ and $\vec{P}$ for $E_2(\vec{P}, \vec{k}, \sigma, r, \lambda)$, and for $E_3(\vec{P}, \vec{k}, \lambda)$, $\vec{d}_{\vec{k},\lambda}(\vec{P})$ explicitly defined in terms of $\vec{k}$ and $\vec{P}$.

A second-order Taylor expansion of the scattering coefficients gives:

Dim-1

$$S_\lambda(k) = \frac{-i\alpha\frac{\lambda}{1+\lambda}}{k + i\frac{1}{2}\alpha\frac{\lambda}{1+\lambda}} \sim -\lambda(1-\lambda)\frac{i\alpha}{k} - \frac{\lambda^2}{2}\frac{\alpha^2}{k^2}$$

Dim-2

$$S_\lambda(k) = \frac{-i\pi}{\frac{1+\lambda}{\lambda}l^{-1} + \gamma + \ln(\frac{k}{2}) - i\frac{\pi}{2}} \sim -\lambda(1-\lambda)i\pi l - i\lambda^2 l^2(\gamma + \ln(\frac{k}{2})) - \frac{\lambda^2}{2}\pi \tag{4.4}$$

Dim-3

$$S_\lambda(k) = \frac{-2ik}{\frac{1+\lambda}{\lambda}l^{-1} + ik} \sim -\lambda(1-\lambda)2ilk - 2\lambda^2 l^2 k^2$$

We can summarize the above expressions as

$$S_\lambda(k) \sim -i\lambda(1-\lambda)c_n k^{n-2} - \frac{\lambda^2}{2}c_n^2 k^{2(n-2)} - \delta_{n,2} i\lambda^2 l^2(\gamma + \ln(\frac{k}{2})),$$

where $c_{\mathbf{1}} = \alpha$, $c_{\mathbf{2}} = \pi l$, and $c_{\mathbf{3}} = 2l$. We will use the following simple lemma.

Lemma 4.1. *Let $k, K \in \mathbb{R}$ and $\vec{k}, \vec{K} \in \mathbb{R}^3$.*

1. *We have the inequality*

$$\frac{1}{\sqrt{(k-\lambda K)^2 + \frac{\alpha^2}{4}\lambda^2}} \leq 2\frac{\sqrt{K^2 + \frac{\alpha^2}{4}}}{\alpha|k|} \leq \frac{2|K| + \alpha}{\alpha|k|},$$

2. *and for dimension one the scattering coefficient satisfies*

$$\left|S_\lambda\left(\left|\frac{k-\lambda K}{1+\lambda}\right|\right)\right| \leq \lambda\frac{2|K| + \alpha}{|k|},$$

3. *and*

$$\left|S_\lambda\left(\left|\frac{k-\lambda K}{1+\lambda}\right|\right) - \frac{-i\alpha\lambda}{|k|}\right| \leq \lambda^2|K|\frac{2|K| + \alpha}{|k|^2}.$$

4. *for dimension three, the scattering coefficient satisfies*

$$\left|S_\lambda\left(\left|\frac{\vec{k}-\lambda\vec{K}}{1+\lambda}\right|\right) - (-2il\lambda|\vec{k}|)\right| \leq \lambda^2\frac{4l}{(1+\lambda)^2}(1 + l|\vec{k}|)(|\vec{k}| + |\vec{K}|).$$

Proof. (1) follows by evaluating the critical points in λ. (2) and (3) follow with an application of (1).

□

Define the following weighted trace norm $\|\cdot\|_{wtn}$ for the density matrices on the single reservoir particle Hilbert space ρ:

$$\|\rho\|_{wtn} = \|\rho\|_1 + \sum_{\epsilon} \sum_{1\leq i,j\leq n} \||\vec{P}|^{n-2+\epsilon}[X_i,[X_j,\rho]]\|_1 \\ + \sum_{\epsilon}\sum_{j=1}^{n} \||\vec{P}|^{n-2+\epsilon}X_j\rho X_j|\vec{P}|^{n-2+\epsilon}\|_1 + \||\vec{P}|^{2(n-2)}\rho|\vec{P}|^{2(n-2)}\|_1, \quad (4.5)$$

where the sums in ϵ are over $\{0,1\}$ for dimension one and $\{-1,0,1\}$ for dimension three. Notice the contrast between dimension $n=1$ and $n=3$ with respect to the weights applied in the norms for the absolute value of the momentum operators $|\vec{P}|$. For $n=1$, $\|\rho\|_{wtn}$ will blow up if ρ has non-zero density of momenta near momentum zero, while for $n=3$, $\|\rho\|_{wtn}$ can blow up if the momentum density does not decay fast enough for large momenta. This difference in requirements for different dimensions can be seen also in the formulas (1.5-1.8). The norm $\|\rho\|_{wtn}$ is not really asymmetric with respect to operators multiplying from the left and the right when ρ is self-adjoint.

Theorem 4.2. *Let $\epsilon(G,\lambda)$ be defined as in (1.1), then there exists a c s.t. for all density operators $\rho\in B_1(L^2(\mathbb{R}^n))$, $G\in B(L^2(\mathbb{R}^n))$, and $0\leq\lambda$*

$$\|\epsilon(G,\lambda)\| \leq c\lambda^3\|\rho\|_{wtn}\|G\|_{wn}. \quad (4.6)$$

Proof. We will prove the result for density operators ρ with a twice continuously differentiable integral kernel $\rho(\vec{k}_1,\vec{k}_2)$ in the momentum representation, and a spectral decomposition $\rho=\sum_{j=1}^{\infty}\lambda_j|f_j\rangle\langle f_j|$ of vectors $f_j(\vec{k})$ that are continuously differentiable in the momentum representation. Since such ρ are dense with respect to the $\|\cdot\|_{wtn}$, the result extends to all ρ with $\|\rho\|_{wtn}<\infty$. By (B.1), the V_1, V_2, $\vec{A}$ operators and the map φ are well defined for all ρ with $\|\rho\|_{wtn}<\infty$ and they vary continuously as a function of ρ with respect to the norm $\|\cdot\|_{wtn}$.

Our challenge is to expand the expressions we found in Propositions 2.2 and 2.3 in λ, until we reach our second-order Taylor expansion while making sure that we only throw away terms which are bounded as in (4.6). We will organize our analysis using the expressions (4.1), (4.2), and (4.3), in conjunction with Propositions 3.1 and 3.3 to effectively transfer the conditions for the boundedness of the differences in our expansions to conditions on ρ and G. Both of the expressions (2.5) and (2.6) have multiple sources of λ dependence. If we expand the expressions involving ρ and f_j first for (2.5) and (2.6) respectively, then the resulting expressions left to expand will be summable in the operator norm and thus not require the heavy preparation involved with the use of Propositions 3.1 and 3.3. Breaking $\mathrm{Tr}_2[\rho\mathbf{S}_\lambda^* G\mathbf{S}_\lambda]$ into parts and dividing by λ we just need bound the differences

$$\frac{1}{\lambda}\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]G-(iV_1+i\lambda V_2+i\frac{\lambda}{2}\{\vec{A},\vec{P}\}-\frac{\lambda}{2}\varphi(I))G, \text{ and} \quad (4.7)$$

$$\frac{1}{\lambda}\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^* G\mathbf{A}]-\lambda\varphi(G), \quad (4.8)$$

where there is a similar expression to (4.7) for $\frac{1}{\lambda}G\mathrm{Tr}_2[\rho\mathbf{A}_\lambda]$. We begin with (4.7), and will have to bound a sequence of intermediate differences. The main differences are the following:

Difference 1

$$\|\frac{1}{\lambda}\mathrm{Tr}_2[\rho\mathbf{A}_\lambda^*]G - \int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\,\tau_{\vec{k}}\,\tau_{\sigma\vec{k}}^*\left(\rho(\vec{k},\sigma\vec{k})+\lambda(\vec{P}+\vec{k})\nabla_T\rho(\vec{k},\sigma\vec{k})\right)\frac{1}{\lambda}\bar{S}_\lambda(|\vec{k}|)G\|,$$

Difference 2

$$\begin{aligned}\|\int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\,\tau_k\,\tau_{\sigma\vec{k}}^*\,(\rho(\vec{k},\sigma\vec{k})+\lambda\big(\vec{P}+\vec{k})\nabla_T\rho(\vec{k},\sigma\vec{k})\big)\\ \big(\frac{1}{\lambda}\bar{S}_\lambda(|\vec{k}|)-\big((1-\lambda)c_n|\vec{k}|^{2-n}+\frac{\lambda}{2}c_n^2|\vec{k}|^{2(2-n)}\big)\big)G\|,\end{aligned}$$

Difference 3

$$\begin{aligned}\|\int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\,\tau_{\vec{k}}\,\tau_{\sigma\vec{k}}^*\left(\rho(\vec{k},\sigma\vec{k})+\lambda(\vec{P}+\vec{k})\nabla_T\rho(\vec{k},\sigma\vec{k})\right)\\ \big((1-\lambda)c_n|\vec{k}|^{2-n}+\frac{\lambda}{2}c_n|\vec{k}|^{2(2-n)}\big)G-(iV_1+i\lambda V_2+\frac{\lambda}{2}\{\vec{A},\vec{P}\}-\frac{\lambda}{2}\varphi(I))G\|.\end{aligned}$$

By the differentiability properties of the integral kernel ρ,

$$\begin{aligned}\rho(\vec{k}+\lambda(\vec{P}+\vec{k}),\sigma\vec{k}+\lambda(\vec{P}+\vec{k}))&=\rho(\vec{k},\sigma\vec{k})+\lambda(\vec{P}+\vec{k})\nabla_T\rho(\vec{k},\sigma\vec{k})\\ &+\lambda^2(\vec{P}+\vec{k})^{\otimes^2}\int_0^1 ds\int_0^s dr\,\nabla_T^{\otimes^2}\rho(\vec{k}+\lambda(\vec{P}+\vec{k})r,\sigma\vec{k}+\lambda(\vec{P}+\vec{k})r),\end{aligned}$$

where $\nabla_T^{\otimes^2}g(x,y)$ is 2 tensor of derivatives with

$$(\nabla_T^{\otimes^2}g(x,y))_{(i,j)}=\lim_{h\to 0}\frac{(\nabla_T g)_i(x+he_j,y+he_j)-(\nabla_T g)_i(x,y)}{h}.$$

The first difference can be rewritten as

$$\begin{aligned}\lambda^2\int_0^1 ds\int_0^s dr\,\|\int_{\mathbb{R}^n} d\vec{k}\int_{SO_n} d\sigma\,\tau_{\vec{k}}\,\tau_{\sigma\vec{k}}^*\,(1+|\vec{P}|)\,(\vec{P}+\vec{k})^{\otimes^2}\,(\delta_{n,3}+|\vec{a}_{\vec{k},r,\lambda}(\vec{P})|^{n-2})\\ \nabla_T^{\otimes^2}\rho(\vec{a}_{\vec{k},r,\lambda}(\vec{P}),\vec{a}_{\vec{k},r,\sigma,\lambda}(\vec{P})+(\sigma-I)\vec{k}))E_1(\vec{P},\vec{k},r,\lambda)G\|.\end{aligned}$$

Using (R1) and expanding the tensor: $(\vec{a}_{\vec{k},r,\lambda}(\vec{P})+\vec{P})^{\otimes^2}$ a single term has the form $\vec{a}_{\vec{k},r,\lambda}(\vec{P})^{\otimes^m}\vec{P}^{\otimes^{2-m}}$. Note that the order of the tensors does not matter in this situation, since the whole vector is in an inner product with $\nabla^{\otimes^2}\rho$, and partial derivatives commute. Now we apply Proposition 3.1 with a single term:

$$\begin{aligned}n_{\vec{k},\sigma}&=E_1(\vec{P},\vec{k},r,\lambda)(\frac{1}{1+r\lambda\sigma})^2\frac{\big(\vec{a}_{\vec{k},r,\lambda}(\vec{P})^{\otimes^m}\otimes\vec{P}^{\otimes^{2-m}}\big)_{j,k}}{|\vec{a}_{\vec{k},r,\lambda}(\vec{P})|^m|\vec{P}|^{2-m}},\\ \eta&=(\delta_{n,3}+|\vec{k}|^{n-2})|\vec{k}|^m(\nabla_T^{\otimes^2}\rho)_{j,k},\\ q_{\vec{k},\sigma}&=n_{\vec{k},\sigma,\lambda}\eta(\vec{a}_{\vec{k},r,\lambda}(\vec{P}),\vec{a}_{\vec{k},r,\sigma,\lambda}(\vec{P})+(\sigma-I)\vec{k}).\end{aligned}$$

Finally with (3.1) we get the bound $\lambda^2 C\|(\delta_{n,3}+|\vec{P}|^{n-2})|\vec{P}|^m(\nabla_T^{\otimes^2}\rho)_{j,k}\|_1\||\vec{P}|^{2-m}(I+|\vec{P}|)G\|$, for some constant C. Note that $\nabla_T\rho = i(\vec{X}\rho-\rho\vec{X})$.

The second difference can be bounded for dimension-one using the inequality

$$|\frac{1}{\lambda}\bar{S}_\lambda(|\vec{k}|)-(\frac{i\alpha(1-\lambda)}{|\vec{k}|}-\frac{\lambda\alpha^2}{2|\vec{k}|^2})|\leq\frac{\lambda^2\alpha^3}{|\vec{k}|^3},$$

and for dimension three using the inequality

$$|\frac{1}{\lambda l}S_\lambda(|\vec{k}|)-(2i(1-\lambda)|\vec{k}|-2\lambda l|\vec{k}|^2)|\leq 2\lambda^2 l^2|\vec{k}|^3.$$

Finally, the last difference comes down to bounding the cross term:

$$\|\int_{\mathbb{R}^n}dk\int_{SO_n}d\sigma\,\tau_{\vec{k}}\,\tau^*_{\sigma\vec{k}}\,\lambda(\vec{P}+\sigma\vec{k})\nabla_T\rho(\vec{k},\sigma\vec{k})\,(\lambda^2 c_n|\vec{k}|^{n-2}+\lambda^2 c_n^2\,|\vec{k}|^{2(n-2)})\,G\|.$$

The bound for the above term follows from (A.1) and that $\int d\vec{k}\,\rho(\vec{k},\vec{k})|\vec{k}|^{2(n-2)}=\||\vec{P}|^{n-2}\rho|\vec{P}|^{n-2}\|_1$.

The $\frac{1}{\lambda}G\mathrm{Tr}_2[\rho\mathbf{A}_\lambda]$ is similarly analyzed so now we study (4.8). Again we have three main differences. There is a λ dependence in $m_{j,\vec{k},\sigma,\lambda}$, $U_{\vec{k},\sigma,\lambda}$, and $S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)$. It is most convenient to begin expanding $m_{j,k,\sigma,\lambda}$ first.

Difference 1

$$\|\frac{1}{\lambda}\mathrm{Tr}_2[\rho\mathbf{A}^*_\lambda G\mathbf{A}_\lambda]-\sum_j\frac{1}{\lambda}\int_{\mathbb{R}^n}d\vec{k}\int_{SO_n\times SO_n}d\sigma_1\,d\sigma_2\,U^*_{\vec{k},\sigma_1,\lambda}\,\det(I+\lambda\sigma_1)^{-\frac{1}{2}}\,\bar{f}_j(\vec{k})$$
$$\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\,G\,S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\,f_j(\vec{k})\,\det(I+\lambda\sigma_2)^{-\frac{1}{2}}\,U_{\vec{k},\sigma_2,\lambda}\|.$$

Difference 2

$$\|\sum_j\int_{\mathbb{R}^n}d\vec{k}\int_{SO_n\times SO_n}d\sigma_1\,d\sigma_2\,U^*_{\vec{k},\sigma_1,\lambda}\,\bar{f}_j(\sigma_1\vec{k})\,\Big[\lambda^{-1}\det(1+\lambda\sigma_1)^{-\frac{1}{2}}\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\,G$$
$$S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\,\det(1+\lambda\sigma_2)^{-\frac{1}{2}}-c_n^2\,|\vec{k}|^{2(n-2)}\,G\Big]\,f_j(\sigma_2\vec{k})\,U_{\vec{k},\sigma_2,\lambda}\|.$$

Difference 3

$$\|\sum_j\lambda^2 c_n^2\int_{\mathbb{R}^3}d\vec{k}\,|f_j(\vec{k})|^2|\vec{k}|^{2(n-2)}\int_{SO_n\times SO_n}d\sigma_1\,d\sigma_2\,(U^*_{\vec{k},\sigma_1,\lambda}\,G\,U_{\vec{k},\sigma_2,\lambda}-\tau^*_{\sigma_1\vec{k}}\,\tau_{\vec{k}}\,G\,\tau^*_{\vec{k}}\,\tau_{\sigma_2\vec{k}})\|.$$

Using the differentiability of f_j's

$$f_j\big(\sigma_1\vec{k}-\lambda\frac{\sigma_1-1}{1+\lambda}(\vec{k}+\vec{P})\big)=f_j\big(\sigma_1\vec{k}\big)+$$
$$\lambda(\frac{\sigma_1-1}{1+\lambda})(\vec{k}+\vec{P})\int_0^1 dr\nabla f_j\big(\sigma_1\vec{k}+r\lambda(\frac{\sigma_1-1}{1+\lambda})(\vec{k}+\vec{P})\big).\quad(4.9)$$

The first difference

$$\|\frac{1}{\lambda}\mathrm{Tr}_2[\rho\mathbf{A}^*_\lambda G\mathbf{A}_\lambda] - \sum_j \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, U^*_{\vec{k},\sigma_1,\lambda}\ \det(1+\lambda\sigma_1)^{-\frac{1}{2}}\ \bar{f}_j(\vec{k})$$
$$\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\, G\, S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|) m_{j,k,\sigma_2,\lambda}(P) U_{\vec{k},\sigma_2,\lambda}\|$$

is less than

$$\lambda^2 c_n^2 \int_0^r dr \| \sum_j \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, U^*_{\vec{k},\sigma_1,\lambda}\ \det(1+\lambda\sigma_1)^{-\frac{1}{2}}$$
$$(\frac{\sigma_1 - I}{I+\lambda})(\mathbf{c}_{\sigma_1,r,\lambda}\vec{v}_{\vec{k},r,\sigma_1,\lambda}(\vec{P}) + \mathbf{c}_{2,\sigma_1,r,\lambda}\vec{P})\frac{\nabla \bar{f}_j(v_{k,\sigma_1,r,\lambda}(P))}{(\delta_{n,3}+|\vec{v}_{\vec{k},r,\sigma_1,\lambda}(\vec{P})|^{n-2})^{-1}}$$
$$E_2(\vec{P},\vec{k},\sigma_1,r,\lambda)(1+|\vec{P}|)\, G\, (1+|\vec{P}|) E_2(\vec{P},\vec{k},\sigma_2,r,\lambda)\, \frac{m_{j,\vec{k},\sigma_2,\lambda}(\vec{P})}{(\delta_{n,3}+|\vec{v}_{\vec{k},r,\sigma_2,\lambda}(\vec{P})|^{n-2})^{-1}}\, U_{\vec{k},\sigma_2,\lambda}\|,$$

where we have rearranged to substitute in the $E_2(\vec{P},\vec{k},\sigma,r,\lambda)$ expressions and used (R3) to rewrite $\vec{k}+\vec{P}$. Two applications of Proposition 3.3 corresponding to $\mathbf{c}_{\sigma_1,r,\lambda}\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P})$ and $\mathbf{c}_{2,\sigma_1,r,\lambda}\vec{P}$ will give us our bound. For the $\mathbf{c}_{\sigma_1,r,\lambda}\vec{v}_{\vec{k},\sigma,r,\lambda}(\vec{P})$ we use Proposition (3.3) with

$$\begin{aligned}
\eta_j^{(1)}(\vec{k}) &= (\delta_{n,3}+|\vec{k}|^{n-2})|\vec{k}||\nabla f_j(\vec{k})|,\\
n^{(1)}_{j,\vec{k},\sigma_1}(\vec{P}) &= \det(I+\lambda\sigma_1)^{-\frac{1}{2}}\, E_2(\vec{P},\vec{k},\sigma,r,\lambda)\, \mathbf{c}_{\sigma_1,r,\lambda}\, \frac{\nabla f_j(\vec{k})}{|\nabla f_j(\vec{k})|},\\
h_{j,\vec{k},\sigma_1} &= n^{(1)}_{j,\vec{k},\sigma_1}\eta_j^{(1)}(\vec{v}_{\vec{k},\sigma_1,\lambda}(\vec{P})),\\
\eta_j^{(2)}(\vec{k}) &= (\delta_{n,3}+|\vec{k}|^{n-2}) f_j(\vec{k}),\\
n^{(2)}_{j,\vec{k},\sigma_2}(\vec{P}) &= \det(I+\lambda\sigma_2)^{-\frac{1}{2}}\, E_2(\vec{P},\vec{k},\sigma_2,r,\lambda),\\
g_{j,\vec{k},\sigma_2} &= n^{(2)}_{j,\vec{k},\sigma_2}\eta_j^{(2)}(v_{k,\sigma_2,\lambda}(P)).
\end{aligned}$$

Hence the term is bounded by a constant multiple of

$$\lambda^2\big(\sum_j \||\vec{P}|(\delta_{n,3}+|\vec{P}|^{n-2})X_j\rho X_j(\delta_{n,3}+|\vec{P}|^{n-2})|\vec{P}|\|_1 + \|(\delta_{n,3}+|\vec{P}|^{n-2})\rho(\delta_{n,3}+|\vec{P}|^{n-2})\|_1\big)\|(1+|\vec{P}|)G(1+|\vec{P}|)\|.$$

The $c_{2,\sigma_1,r,\lambda}P$ term is bounded by a constant multiple of

$$\lambda^2\big(\sum_j \|(\delta_{n,3}+|\vec{P}|^{n-2})X_j\rho X_j(\delta_{n,3}+|\vec{P}|^{n-2})\|_1 + \|(\delta_{n,3}+|\vec{P}|^{n-2})\rho(\delta_{n,3}+|\vec{P}|^{n-2})\|_1\big)\|(1+|\vec{P}|)|\vec{P}|G(1+|\vec{P}|)\|.$$

The next intermediary difference has the form:

$$\|\sum_j \int_{\mathbb{R}^n} dk \int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, U^*_{\vec{k},\lambda,\sigma_1}\ \det(1+\lambda\sigma_1)^{-\frac{1}{2}}\ \bar{f}_j(\vec{k})\ \bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)$$
$$G\,(f(\vec{v}_{\vec{k},\sigma_2,\lambda}(\vec{P})) - f(\sigma_2\vec{k}))\ \bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)\det(1+\lambda\sigma_2)^{-\frac{1}{2}}\, U_{\vec{k},\lambda,\sigma_1}\|.$$

Expanding $f(\vec{v}_{\vec{k},\sigma_2,\lambda}(\vec{P})) - f(\sigma_2\vec{k})$ as in (4.9), we can apply a similar analysis to the above, except that for the left-hand side we organize around $E_3(\vec{P},\vec{k},\lambda)$ rather than $E_3(\vec{P},\vec{k},\sigma,r,\lambda)$.

Due to $\bar{f}_j(\sigma_1\vec{k})f_j(\sigma_2\vec{k})$, the second difference is summable, and we do not need to prepare any more applications of Proposition 3.3. We begin by bounding

$$\|\sum_j \lambda f\, c_n^2 \int_{\mathbb{R}^n} d\vec{k} \int_{SO_n\times SO_n} d\sigma_1\, d\sigma_2\, U^*_{\vec{k},\sigma_1,\lambda} \det(1+\lambda\sigma_1)^{-\frac{1}{2}} \frac{\bar{f}_j(\sigma_1\vec{k})}{|\vec{k}|^{2-n}} \Big[\frac{|k|^{2-n}}{\lambda c_n} \bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)G$$
$$\frac{|k|^{2-n}}{\lambda c_n} S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|) - G\Big] \frac{f_j(\sigma_2\vec{k})}{|\vec{k}|^{2-n}} \det(1+\lambda\sigma_2)^{-\frac{1}{2}} U_{\vec{k},\sigma_2,\lambda}\|.$$

We observe the inequality

$$\|\frac{|\vec{k}|^{2-n}}{\lambda c_n}\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|)G\frac{|\vec{k}|^{2-n}}{\lambda c_n}S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|) - G]\| \leq$$
$$\frac{1}{c_n}\|\big(\frac{|\vec{k}|^{2-n}}{\lambda c_n}\bar{S}_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|) - i\big)G(|\vec{P}|+I)E_3(\vec{P},\vec{k},\lambda)\|$$
$$+ \|G\big(\frac{|\vec{k}|^{2-n}}{\lambda c_n}S_\lambda(|\vec{d}_{\vec{k},\lambda}(\vec{P})|) + i\big)\|.$$

By (3) and (4) of (4.1), the right-hand side is bounded by a sum of terms proportional to $\lambda|\vec{k}|^{r(n-2)}\||\vec{P}|^{\epsilon_1}G(I+|\vec{P}|)^{\epsilon_2}\|$ for $r = 0,1,2$, $\epsilon_1,\epsilon_2 = 0,1$. Bounding the above integral is then routine and requires that $\||\vec{P}|^{2(n-2)}\rho|\vec{P}|^{2(n-2)}\|_1$. The last thing to do for the second difference is expanding $\det(1+\lambda\sigma_1)^{-\frac{1}{2}}$ and $|\det(1+\lambda\sigma_2)|^{-\frac{1}{2}}$, which does not pose much difficulty.

For the third difference, we will need to work with the $D_{\frac{1+\lambda\sigma}{1+\lambda}}$ term.

$$\|U^*_{\vec{k},\sigma_1,\lambda}\, G\, U_{\vec{k},\sigma_2,\lambda} - \tau^*_{\sigma_1\vec{k}}\,\tau_{\vec{k}}\, G\, \tau^*_{\vec{k}}\,\tau_{\sigma_2\vec{k}}\| \leq \|(D^*_{\frac{1+\lambda\sigma_1}{1+\lambda}} - I)G\| + \|G(D_{\frac{1+\lambda\sigma_1}{1+\lambda}} - I)\|,$$

since $U_{\vec{k},\sigma_2,\lambda}$, $\tau_{\sigma\vec{k}}$, and $\tau_{\vec{k}}$ are unitary and $D^*_{\frac{1+\lambda\sigma_1}{1+\lambda}}\tau_{\vec{k}} = \tau_{\frac{1+\lambda\sigma}{1+\lambda}k}D^*_{\frac{1+\lambda\sigma_1}{1+\lambda}}$. $D_{\frac{1+\lambda\sigma}{1+\lambda}}$ satisfies the integral relation

$$D_{\frac{1+\lambda\sigma}{1+\lambda}} = I + \int_0^\lambda ds\,\{\frac{d}{ds}\log(\frac{1+s\sigma}{1+s})D_{\frac{1+s\sigma}{1+s}}\vec{P},\vec{X}\},$$

and hence

$$\|\frac{1}{\lambda}(D_{\frac{1+\lambda\sigma}{1+\lambda}} - I)G\| \leq (\sup_{0\leq s\leq\lambda}\frac{d}{ds}\log(\frac{1+s\sigma}{1+s}))\|\sum_{i,j}\|P_iX_jG\|.$$

The third difference is then bounded by a fixed constant multiple of $\lambda^2\||\vec{P}|^{n-2}\rho|\vec{P}|^{n-2}\|_1\sum_j\||\vec{P}|X_jG\|$. □

APPENDIX

A Hilbert spaces and operator inequalities

Lemma A.1. *Let η be a trace class operator on $\mathbb{R}^n$ with continuous integral kernel $\eta(\vec{k}_1,\vec{k}_2)$, $A, A' \in GL_n(\mathbb{R})$, and $\vec{a},\vec{a}' \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} d\vec{x}\,|\eta(A\vec{x}+\vec{a}, A'\vec{x}+\vec{a}')| \leq \frac{1}{2}(\frac{1}{|\det(A)|} + \frac{1}{|\det(A')|})\|\eta\|_1.$$

Proof. Let $\rho = \sum_j \lambda_j |f_j\rangle\langle g_j|$, then we have

$$\int_{\mathbb{R}^n} d\vec{x}\, |\rho(A\vec{x}+\vec{a}, A'\vec{x}+\vec{a}')| = \int_{\mathbb{R}^n} d\vec{x}\, |\sum_j \lambda_j f_j(A\vec{x}+\vec{a})\, \bar{g}_j(A'\vec{x}+\vec{a}')|$$
$$\leq \frac{1}{2}(\frac{1}{|\det(A)|} + \frac{1}{|\det(A')|})\|\rho\|_1, \quad \text{(A.1)}$$

where the inequality follows from $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}^+$ and completing the integration. The equality on the left-hand side of (A.1) is formal for a general integral kernel $\rho(\vec{k}_1, \vec{k}_2)$ which is defined only a.e. with respect to joint integration over $\vec{k}_1, \vec{k}_2$, but with our continuity condition it is well defined.

□

Proposition A.2. *For $n \in \mathbb{N}$, let $A_n \in B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ and*

$$\frac{1}{2}\sum_n |A_n| + |A_n^*|$$

be weakly convergent to a bounded operator with norm c. Then $\sum_n A_n$ is strongly convergent to a bounded operator X with $\|X\| \leq c$.

Proof. Let $g \in \mathcal{H}$ and with the polar decomposition [11] $A_n = U_n|A_n|$, then taking a tail sum

$$\|\sum_N^\infty A_n g\|_2 = \sup_{\|h\|_2=1} \langle h| \sum_N^\infty U_n |A_n|^{\frac{1}{2}} |A_n|^{\frac{1}{2}} g\rangle$$
$$\leq \sup_{\|h\|_2=1} \sum_N^\infty \||A_n|^{\frac{1}{2}} U_n h\|_2 \||A_n|^{\frac{1}{2}} g\|_2 \leq \sup_{\|h\|_2=1} \big(\sum_{n=1}^\infty \langle h||A_n^*|h\rangle\big)^{\frac{1}{2}} \big(\sum_N^\infty \langle g||A_n|g\rangle\big)^{\frac{1}{2}},$$

where the first and second inqualities follow by two different applications of the Cauchy-Schwartz inequality. The right-hand side then tends to zero for large N by our assumptions on the series $\sum |A_n|$ and $\sum_n |A_n^*|$. The operator norm bound can be seen from the same calculation with a sum over all n rather than a tail.

□

Proposition A.3. *Let A_n, B_n for $n \in \mathbb{N}$ be elements in $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ such that $\sum_n A_n^* A_n$ and $\sum_n B_n^* B_n$ converge weakly to bounded operators with norms less than c, then the sum*

$$\varphi(G) = \sum_n A_n^*\, G\, B_n$$

is strongly convergent to an operator with norm less than or equal to $c\|G\|$.

The proof follows a similar pattern to that of Proposition A.2.

B Norm bounds for V_1, V_2, $\vec{A}$ and φ

The following lemma shows that the limiting expressions (1.3) vary continuously with respect to the density operator ρ in the $\|\cdot\|_{wtn}$ topology. It allows the limiting expression to be defined for all ρ with $\|\rho\|_{wtn} < \infty$ without the additional assumption that the integral kernel of ρ in the momentum representation in continuously differentiable. A somewhat weaker norm than $\|\cdot\|_{wtn}$ would suffice.

Lemma B.1. *Let V_1, V_2, $\vec{A}$, and φ be defined as in (1.5)-(1.8) for a $\rho \in \mathcal{B}_1(\mathbb{R}^n)$, $\rho \geq 0$, with continuously differentiable integral kernel in the momentum representation, then there is a constant $c > 0$ such that for all ρ and j*

$$\|V_1\|, \|V_2\|, \|\vec{A}\|, \|[P_j, A_j]\|, \|\varphi\| \leq c\|\rho\|_{wtn}.$$

Proof. By an argument similar to (A.1)

$$\|V_1\| \leq c_n \||\vec{P}|\rho\|_1, \qquad \|\vec{A}\| \leq c_n \sum_j \||\vec{P}|^{n-2} X_j \rho\|_1, \qquad \|\varphi\| \leq c_n^2 \||\vec{P}|^{n-2}\rho|\vec{P}|^{n-2}\|_1,$$

$$\|V_2\| \leq c_n \sum_j \|\{P_j|\vec{P}|^{n-1}, [X_j, \rho]\}\|_1, \qquad \|[P_j, A_j]\| \leq c_n \|[P_j|\vec{P}|^{n-2}, [X_j, \rho]]\|_1,$$

where we have used that $\|\varphi\| = \|\varphi(I)\|$ since φ is a positive map. By ρ being self-adjoint, we have inequalities such as

$$\|P_j|\vec{P}|^{n-2}\rho X_j\|_1 \leq \frac{1}{2}(\|X_j\rho X_j\|_1 + \|P_j|\vec{P}|^{n-2}\rho|\vec{P}|^{n-2}P_j\|_1),$$

and then that $P_j \leq |\vec{P}|$. Finally, since ρ is positive $\int_{\mathbb{R}^n} d\vec{k}\rho(\vec{k}, \vec{k})|\vec{k}|^{2s} = \||\vec{P}|^s\rho|\vec{P}|^s\|_1$, and inequalities of the form

$$\||\vec{P}|^r\rho|\vec{P}|^r\|_1 \leq \|\rho\|_1 + \||\vec{P}|^s\rho|\vec{P}|^s\|_1$$

follow, where r, s have the same sign and $|r| \leq |s|$. Hence $\|\rho\|_{wtn}$ bounds the expressions for V_1, V_2, $\vec{A}$, $[P_j, A_j]$ and φ.

□

Acknowledgments

I thank Bruno Nachtergaele for discussions on this manuscript. This work is partially funded from the Belgian Interuniversity Attraction Pole P6/02. Also financial support has come from Graduate Student Research (GSR) fellowships funded by the National Science Foundation (NSF # DMS-0303316 and DMS-0605342).

References

[1] R. Adami, R. Figari, D. Finco, A. Teta: *On the Asymptotic Dynamics of a Quantum System composed by Heavy and Light Particles*, Comm. Math. Phys. **268**, 819–852 (2006).

[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, Springer-Verlag, Berlin, (1988).

[3] C. Cacciapuoti, R. Carlone, R. Figari: *Decoherence Induced by Scattering: a Three-Dimensional Model*, J. Phys. **A38**, 4933–4946 (2005).

[4] J. Clark: *An infinite-temperature limit for a scattering process*, Rept. Math. Phys. (2009), in press, arXiv 0801.0722.

[5] D. Dürr, R. Figari, A. Teta: *Decoherence in a Two Particle Model*, J. Math. Phys. **45**, 1291-1309 (2004).

[6] M. R. Gallis, G. N. Fleming: *Environmental and Spontaneous Localization*, Phys. Rev. **A42**, 38-48 (1990).

[7] L. Hackermüller, K. Hornberger, B. Brezger, A. Zeilinger, M. Arndt: *Decoherence in a Talbot-Lau Interferometer: the Influence of Molecular Scattering*, Appl. Phys. **B77**, 781–787 (2003).

[8] K. Hornberger, J. Sipe: *Collisional Decoherence Reexamined*, Phys. Rev. **A68**, 012105 (2003).

[9] E. Joos, H. D. Zeh: *The Emergence of Classical Properties Through Interaction with the Environment*, Z. Phys. **B59**, 223–243 (1985).

[10] G. Lindblad: *On the Generators of Quantum Dynamical Semigroups*, Comm. Math. Phys. **48**, 119–130 (1976).

[11] M. Reed, B. Simon: *Functional Analysis*, Academic Press, (1980).